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Norm-Attaining Operators into Strictly Convex Banach Spaces

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If a strictly convex Banach space Y contains either a symmetric basic sequence which is not equivalent to the l_1-basis or a normalized sequence with an upper p-estimate, then there is a Banach space X such that the set of norm-attaining operators is not dense in the Banach space of all bounded linear operators from X into Y. We deduce that no infinite-dimensional uniformly convex Banach space has Lindenstrauss' property B. © 1998 Academic Press

INTRODUCTION

The Bishop–Phelps theorem, the origin of the so-called "perturbed optimization principles," asserts that the set of norm-attaining functionals on a Banach space is dense in the dual space for the norm topology. In their seminal paper, Bishop and Phelps [3] addressed the question of what Banach spaces might play the role of the scalar field in their theorem.

Intensive research about this question was initiated in 1963 by Lindenstrauss [12], who introduced the so-called property B. Given Banach spaces X and Y, let us consider the Banach space $L(X,Y)$ of bounded linear operators from X into Y and let us denote by $NA(X,Y)$ the set of norm-attaining operators; that is, $T \in NA(X,Y)$ if there is $x \in B_X$ (the unit ball of X) such that $\|Tx\| = \|T\|$. The Banach space Y is said to satisfy property B if $NA(X,Y)$ is dense in $L(X,Y)$ for all Banach spaces X. Positive results about property B can be found in [12, 14, 2].

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Concerning negative results, most known examples of Banach spaces without property B are strictly convex. The first one is due to Lindenstrauss: A strictly convex Banach space fails B as soon as there is a noncompact operator from c_0 into it. In the same paper [12] Lindenstrauss also proved that if X and Y are isomorphic, B_X lacks extreme points, and Y is strictly convex, then $NA(X, Y)$ is not dense in $L(X, Y)$. For instance, every strictly convex Banach space isomorphic to $L_1[0, 1]$ fails property B. Actually, it follows from a result by Uhl [16] that a strictly convex Banach space without the Radon–Nikodym property cannot have property B. However, the most striking negative result was obtained in 1990 by Gowers [9]: For $1 < p < \infty$, l_p fails property B.

In view of such an amount of examples, one is tempted to conjecture that, at least in the infinite-dimensional context, property B and strict convexity are incompatible. Our more general result, which seems to support this idea, reads that a strictly convex Banach space fails property B if there is a noncompact operator from any member in a certain family of preduals of Lorentz sequence spaces into it. No matter how unpleasant this assumption may look, it is very frequent. We show that it is fulfilled by any Banach space containing either a symmetric basic sequence which is not equivalent to the l_1-basis or a normalized sequence with an upper p-estimate. As a consequence, we get the main result of this paper: Every infinite-dimensional uniformly convex space fails property B. Actually, we prove that a strictly convex Banach space containing an infinite-dimensional super-reflexive subspace cannot have property B.

RESULTS

Let us start by recalling the definition of a family of Banach spaces whose relevance in the theory of norm-attaining operators was first observed by Gowers. This family has been recently used to solve some problems related to norm-attaining multilinear forms and polynomials [11].

By *admissible sequence* w we shall mean a decreasing sequence $w = (w(n))$ of positive numbers such that $w(1) = 1$ and $w \in c_0 \setminus l_1$. If w is an admissible sequence, the Banach space $d_*(w)$ is defined by

$$d_*(w) = \left\{ x \in c_0 \colon \lim_{n \to \infty} \frac{\sum_{k=1}^{n} \tilde{x}(k)}{\sum_{k=1}^{n} w(k)} = 0 \right\},$$

where $(\tilde{x}(n))$ is the decreasing rearrangement of $(|x(n)|)$. The norm of $d_*(w)$ is given by

$$\|x\| := \sup_n \frac{\sum_{k=1}^{n} \tilde{x}(k)}{\sum_{k=1}^{n} w(k)}.$$

It is well known [8, 15] that $d_*(w)$ is a predual of the Lorentz sequence space $d(w,1)$, the Banach space of all sequences of scalars $x = (x(n))$ for which

$$\|x\| := \sup_{\pi} \sum_{k=1}^{\infty} |x(\pi(n))| w(n) < \infty,$$

where π ranges over all permutations of the integers [13]. In fact, $d_*(w)$ is an M-ideal in its bidual and is the only predual of $d(w,1)$ with this property [17, 10].

In [9], Gowers proved that the unit ball of $d_*(w)$ lacks extreme points. Actually, the following lemma shows that faces in the unit ball of $d_*(w)$ look very much like those of c_0. The proof can be found in [1] (for the special case $w(n) = 1/n$) and in [11]. Throughout, the unit vector basis of $d_*(w)$ will be denoted by (e_n).

LEMMA 1. *For every $x \in d_*(w)$ with $\|x\| = 1$, there exist a natural number m and $\delta > 0$ such that $\|x + \lambda e_k\| \leq 1$ for $k \geq m$ and any scalar λ with $|\lambda| \leq \delta$.*

Now, we are in position to exploit with the spaces $d_*(w)$ the same argument used by Lindenstrauss with c_0. Let Y be a strictly convex Banach space, $T \in NA(d_*(w), Y)$, and $x \in B_{d_*(w)}$ such that $\|Tx\| = \|T\|$. If m and δ are given by the above lemma, we have $\|T(x \pm \delta e_n)\| \leq \|T\| = \|Tx\|$ for $n \geq m$, and the strict convexity of Y gives $Te_n = 0$. We have showed:

PROPOSITION 2. *Every norm-attaining operator from $d_*(w)$ into a strictly convex Banach space is a finite-rank operator.*

COROLLARY 3. *If Y is a strictly convex Banach space and for some admissible sequence w there is a noncompact operator from $d_*(w)$ into Y, then Y fails property* B.

We present two ways to derive profit from this corollary. The first one is a slight improvement of a result by Gowers. In [9], Gowers shows that, by taking $w = (1/n)$, $d_*(w)$ is contained (as a vector space) in l_p for $1 < p < \infty$ and the formal identity is a bounded operator from $d_*(w)$ to l_p which is far from any norm-attaining operator. Getting the most out of Gowers' arguments, one can easily prove that a strictly convex Banach space which contains an isomorphic copy of l_p, $1 < p < \infty$, cannot have property B. The possibility of using any noncompact operator will allow a further improvement, due to the following technical result:

Proposition 4. *Let Y be a Banach space containing a normalized, symmetric basic sequence (y_n) which is not equivalent to the unit vector basis in l_1. Then there is an admissible sequence w and an operator $T \in L(d_*(w), Y)$ such that*

$$T(e_n) = y_n, \qquad \forall n \in \mathbb{N}$$

(in particular, T is not compact). If (y_n) is (equivalent to) the unit vector basis of l_p with $1 < p < \infty$, any admissible sequence $w \in l_p$ works.

The proof requires the following lemma:

Lemma 5 [15]. *Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be real numbers satisfying*

$$\sum_{k=1}^{m} a_k \leq \sum_{k=1}^{m} b_k \qquad \textit{for } m = 1, 2, \dots, n.$$

If $c_1 \geq c_2 \geq \cdots \geq c_n \geq 0$, then

$$\sum_{k=1}^{n} a_k c_k \leq \sum_{k=1}^{n} b_k c_k.$$

Proof of Proposition 4. Since (y_n) is not equivalent to the l_1-basis, there exists a series $\sum_{n=1}^{\infty} w(n) y_n$ in Y which converges unconditionally but not absolutely. Let us show that the sequence $w = (w(n))$ can be taken to be admissible. Since (y_n) is unconditional we may arrange $w(n) \geq 0$ for all n. Actually, we may have $w(n) > 0$ since (y_n) is subsymmetric (i.e., equivalent to any subsequence). Finally, the symmetry of (y_n) allows rearranging w to get a decreasing sequence. Thus, we have an admissible sequence w such that $\sum_{n \geq 1} w(n) y_n$ converges. Clearly, if (y_n) is equivalent to the l_p-basis with $1 < p < \infty$, we can use any admissible sequence $w \in l_p$.

Now, for $x \in B_{d_*(w)}$ with finite support, that is, $x = \sum_{k=1}^{n} x(k) e_k$, we define

$$T(x) = \sum_{k=1}^{n} x(k) y_k$$

and we are left with showing that T is bounded on this kind of sequence.

Fix $x = \sum_{k=1}^{n} x(k) e_k$ with $\|x\| \leq 1$ and let y^* be an arbitrary functional of the unit ball of Y^*. Consider a permutation π of the integers satisfying that $\pi(k) \leq n$ for $k \leq n$, and $|y^*(y_{\pi(1)})| \geq |y^*(y_{\pi(2)})| \geq \cdots \geq |y^*(y_{\pi(n)})|$. Finally, choose scalars μ_k such that $|\mu_k| = 1$ and $|y^*(y_{\pi(k)})| = \mu_k y^*(y_{\pi(k)})$ for $k \leq n$.

Then we have

$$\begin{aligned}|y^*(Tx)| &= \left|\sum_{k=1}^{n} x(k)y^*(y_k)\right|\\ &\le \sum_{k=1}^{n} |x(k)|\,|y^*(y_k)|\\ &= \sum_{k=1}^{n} |x(\pi(k))|\,|y^*(y_{\pi(k)})|.\end{aligned}$$

Since $\|x\| \le 1$ we have $\sum_{k=1}^{m}|x(\pi(k))| \le \sum_{k=1}^{m} w(k)$ for $m \le n$ and the previous lemma gives

$$\begin{aligned}|y^*(T(x))| &\le \sum_{k=1}^{n} w(k)|y^*(y_{\pi(k)})|\\ &= \sum_{k=1}^{n} w(k)\mu_k y^*(y_{\pi(k)}) = y^*\left(\sum_{k=1}^{n} w(k)\mu_k y_{\pi(k)}\right)\\ &\le \left\|\sum_{k=1}^{n} w(k)\mu_k y_{\pi(k)}\right\| \le K\left\|\sum_{k=1}^{n} w(k)y_k\right\| \le KM,\end{aligned}$$

where K is the symmetric constant of (y_n) and

$$M = \sup\left\{\left\|\sum_{k=1}^{n} w(k)y_k\right\| : n \in \mathbb{N}\right\}.$$

Since $y^* \in B_{Y^*}$ is an arbitrary functional we have

$$\|Tx\| \le KM$$

as required. ∎

By linking Corollary 3 and the above proposition, we have:

COROLLARY 6. *If Y is a strictly convex Banach space containing a symmetric basic sequence which is not equivalent to the unit vector basis in l_1, then Y fails property* B.

This corollary includes, as special cases, the above-mentioned examples by Lindenstrauss and Gowers: A strictly convex Banach space containing a subspace isomorphic to c_0 or l_p with $1 < p < \infty$ cannot have property B. This is not too surprising since we generalized the argument of Gowers to prove Corollary 6. Let us also mention that, as shown by Figiel and Johnson [7] and Altshuler (see [13, Example 3.b.10]), there are Banach spaces with symmetric basis which contain no subspace isomorphic to c_0 or l_p, $p \geq 1$. No strictly convex renorming of these spaces has property B.

Our second way to exploit Corollary 3 is even simpler than the first one.

PROPOSITION 7. *Let Y be a Banach space which contains a normalized sequence (y_n) with an upper p-estimate for some $p > 1$; that is, there exists $M > 0$ such that*

$$\left\| \sum_{k=1}^{n} a_k y_k \right\| \leq M \left(\sum_{k=1}^{n} |a_k|^p \right)^{1/p} \qquad (*)$$

for all $n \in \mathbb{N}$ and every scalar sequence (a_k). Then, by taking $w(n) = 1/n$ $\forall n \in \mathbb{N}$, there exists a noncompact operator from $d_*(w)$ into Y.

Proof. Let us recall [9] that $d_*(w)$ (with $w(n) = 1/n$, $\forall n \in \mathbb{N}$) is contained in l_p and the formal identity is a bounded operator from $d_*(w)$ into l_p. Condition $(*)$ clearly gives a bounded linear operator from l_p into Y taking the l_p-basis into the sequence (y_n). By composing both operators we get $T \in L(d_*(w), Y)$ such that $Te_n = y_n$ for all n. Since (y_n) converges weakly to 0 and $\|y_n\| = 1$ for all n, T is not compact. ∎

A strictly convex Banach space satisfying the assumption in the above proposition cannot have property B. For example, Tsirelson's original space contains no symmetric basic sequence but contains sequences with upper p-estimates for all $p > 1$ [4, Prop. V.10] so every strictly convex renorming of this space fails property B. The following consequence of the above results is more interesting.

THEOREM 8. *A strictly convex Banach space which contains an infinite-dimensional super-reflexive subspace cannot have property* B.

Proof. Let Z be a super-reflexive subspace of a strictly convex Banach space Y. As shown by Enflo [6], Z is uniformly convexifiable and the theorem of N. and V. Gurarii (see, e.g., [5, Theorem VIII.3]) gives that any normalized basic sequence in Z has an upper p-estimate. Now apply Proposition 7 and Corollary 3. ∎

The following special case of the above theorem is probably the main result of this paper.

COROLLARY 9. *Every infinite-dimensional uniformly convex Banach space fails property* B.

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